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MOTIONS DOUBLY ASYMPTOTIC TO INVARIANT TORI IN THE THEORY OF PERTURBED HAMILTONIAN SYSTEMS*

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Poincaré's theory /1/ of the formation of isolated periodic motions during the perturbation of resonant invariant tori of integrable Hamiltonian systems was generalized in /2/ by the methods of KAM-theory to the case of conditionally-periodic motions. In this paper variational methods are used to prove the existence of motions doubly-asymptotic to the nascent invariant tori. The existence of such trajectories is important in the qualitative investigation of a perturbed system. For example, if the doubly-asymptotic trajectory is isolated, then the perturbed system is non-integrable /3/ and possesses stochastic behaviour. Arnol'd's /4/ diffusion for Hamiltonian systems with many degrees of freedom is based on the existence of motions doubly-asymptotic to invariant tori.

Let the Hamiltonian function $H = H_0 + \varepsilon H_1 + O(\varepsilon^2)$ of an autonomous Hamiltonian system with m degrees of freedom depends smoothly on the parameter ε. We assume that the unperturbed system with Hamilton function H_0 has a smooth compact invariant m-dimensional Lagrangian manifold M (a manifold M is Lagrangian if the restriction to M of the phase space's canonical 2-form is zero), entirely filled with n-dimensional invariant tori carrying conditionally-periodic motions with identical vector frequencies $\omega \in \mathbf{R}^n$. This means that a free action of the n-dimensional torus $\mathbf{T}^n = \mathbf{R}^n/2\pi Z^n$ is specified on M: $\varphi \in \mathbf{T}^n,\ x \in M \to f(\varphi, x) \in M$, and for any $x \in M$ the curve $t \to f(\omega t, x)$ is a trajectory of the unperturbed Hamiltonian system. A principal example is the case /2/ when the unperturbed system is fully integrable, and M is its m-dimensional resonant torus, such that the corresponding frequency vector $\Omega \in \mathbf{R}^m$ satisfies $m - n$ resonance relations of the form $\langle k, \Omega\rangle = 0,\ k \in \mathbf{Z}^m$.

A neighbourhood of the Lagrangian manifold M in the phase space can be identified with a neighbourhood of the set $\{y = 0\}$ in the cotangent bundle $T^*M = \{(x, y): x \in M, y \in T_x{}^*M\}$ with canonical 2-form $dx \wedge dy$. We extend the action of the torus T^n on M to the Hamiltonian action of $\mathbf{T}^n$ on T^*M:

$$\varphi \in \mathbf{T}^n,\ (x, y) \in T^*M \to (f(\varphi, x),\ f_x{}^{*-1}y) \tag{1}$$

Let $\bar{H}_0$ and $\bar{H}_1$ be the results of averaging the functions H_0 and H_1 with respect to the action (1), for example

$$\bar{H}_0(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbf{T}^n} H_0(f(\varphi, x),\ f_x{}^{*-1}y)\, d\varphi \tag{2}$$

We make the following assumptions:

1) the frequency vector ω is non-resonant in the sense of KAM-theory: there exist $C > 0$ and $N > n - 1$ such that

$$|\langle \omega, k\rangle| \geqslant C \| k \|^{-N} \tag{3}$$

for all non-zero $k \in \mathbf{Z}^n$.

2) the following convexity constraint is satisfied: the Hessian $A(x) = \bar{H}_{0yy}(x, 0)$ is positive-definite for all $x \in M$. This condition can be weakened, for example, by changing it to the

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isoenergetic convexity condition

$$\langle A(x)\xi, \xi\rangle \geqslant c\|\xi\|^2, \quad c > 0$$

for all ξ such that $\langle \xi, \bar{H}_{0y}(x, 0)\rangle = 0$, and all $x \in M$.

The function $V(x) = \bar{H}_1(x, 0)$ on M is invariant with respect to the action of the torus T^n and reaches its maximum on some torus $\Gamma_0 = \{f(\varphi, x_0): \varphi \in \mathbf{T}^n\}$.

3) we assume that this maximum is strict and non-degenerate, i.e. the rank $V_{xx}(x_0)$ is equal to $m-n$.

One can then show that for sufficiently small $\varepsilon > 0$ the perturbed system has a smooth invariant torus $\Gamma = \{g(\varphi, \sqrt{\varepsilon}): \varphi \in T^n\}$ in phase space, smoothly depending on $\sqrt{\varepsilon}$, coinciding with Γ_0 when $\varepsilon = 0$ and filled with conditionally-periodic motions $t \to g(\omega t + \varphi_0, \sqrt{\varepsilon})$ with frequency vector ω. Through Γ there pass two invariant m-dimensional Lagrangian manifolds $\Lambda^{\pm}$, depending smoothly on $\sqrt{\varepsilon}$, coinciding with M for $\varepsilon = 0$, and intersecting along Γ at a non-zero angle of order $\sqrt{\varepsilon}$ and filled with trajectories of the perturbed system, asymptotic to Γ as $t \to \pm\infty$ respectively.

For $n = 1$, when the manifold M is filled with periodic trajectories of the unperturbed system, these assertions are due to Poincaré /1/. In the general case they are a reformulation of some of the results in /2/. The proof is performed using the methods of /5/. The case when M is a resonant invariant torus of a completely integrable system was considered in /2/, and the version given here does not require a different proof. Instead of condition 2) the weaker condition of non-degeneracy was used. Assumption 2) is necessary to prove the following theorem.

Theorem. For sufficiently small $\varepsilon > 0$ there exists a trajectory of the perturbed system, doubly-asymptotic (homoclinic) to Γ as $t \to \pm\infty$ and contained in a neighbourhood of M with of order $\sqrt{\varepsilon}$. In particular, $(\Lambda^+ \cap \Lambda^-) \setminus \Gamma \neq \varnothing$.

Generally speaking, this trajectory is doubly-asymptotic to various conditionally-periodic motions on the torus Γ as $t \to \pm\infty$. In the same way it can be shown that for all $x \in M$ there exists a trajectory asymptotic to Γ, whose projection into M passes through x, i.e. $\pi\Lambda^{\pm} = M$, where $\pi: T^*M \to M$ is the projection. In certain cases one can give estimates of the number of doubly-asymptotic trajectories to Γ. For example, if $M = \mathbf{T}^m$ is a resonant torus, then their number is at least $m-n$.

The proof of the theorem is based on the methods described in /6/. From Taylor's formula

$$H = \langle v(x), y\rangle + {}^1/_2\langle H_{0yy}(x, 0)\, y, y\rangle + \varepsilon H_1(x, 0) + O(\varepsilon^2 + \varepsilon|y| + |y|^3) \tag{4}$$

apart from a constant.

The trajectories of the vector field $v(x) = H_{0y}(x, 0)$ of the unperturbed system on M have the form $t \to f(\omega t, x)$. Hence $v(x) = f_\varphi(0, x)\,\omega$ and $\langle v(x), y\rangle = \langle \omega, I\rangle$, where $I = f_\varphi^*(0, x)\, y$ is the momentum associated with the Hamiltonian action (1) of the torus $\mathbf{T}^n$ on T^*M. (In classical terminology I is the action corresponding to the angular variable $\varphi \in \mathbf{T}^n$).

Because the frequency vector ω is non-resonant (3), one can change the variables, representing a step of the classical method of averaging "over the fast variable $\varphi \in \mathbf{T}^n$", reducing the Hamiltonian function (4) to the form /2/

$$\bar{H} = \langle \omega, I\rangle + {}^1/_2\langle \bar{H}_{0yy}(x, 0)\, y, y\rangle + \varepsilon H_1(x, 0) + O(\varepsilon^2 + \varepsilon|y| + |y|^3) \tag{5}$$

Here $\bar{H}_0$ is the function (2) and the transformed variables are denoted by unchanged letters.

From condition (2) the positive-definite quadratic form $\|y\|^2 = \langle A(x)\, y, y\rangle$, $y \in T_x^*M$ defines a Riemannian metric on M. From (5), for sufficiently small $\varepsilon > 0$ and $\delta > 0$ the function $\bar{H}$ is strongly convex with respect to y in the domain $\{(x, y): \|y\|^2 \leqslant \delta\}$:

$$a\|\xi\|^2 \leqslant \langle \bar{H}_{yy}\xi, \xi\rangle \leqslant b\|\xi\|^2 \tag{6}$$

for all ξ, where the constants $0 < a < b$ do not depend on ε. We redefine the Hamiltonian function (5) inside the domain $\{\|y\|^2 \leqslant \delta\}$ so that it smoothly depends on ε and so that for all sufficiently small $\varepsilon > 0$ and all $\xi \in T_x^*M$ it satisfies the inequality (6), (which is possible, with changed a and b), and is identical with a second-degree polynomial in y as $\|y\| \to \infty$. It will be shown that for sufficiently small $\varepsilon > 0$ the constructed Hamiltonian system has a trajectory doubly-asymptotic to Γ, entirely contained in the region $\{\|y\|^2 \leqslant C\varepsilon\}$. Here and below A, B and C are positive constants independent of ε. For $\varepsilon < \delta / C$ this proves the theorem.

We perform the canonical transformation $y \mapsto \eta = y / \sqrt{\varepsilon}$, $\bar{H} \mapsto F = \bar{H} / \sqrt{\varepsilon}$. The Hamilton's function then takes the form

$$F = \langle v(x), \eta\rangle + \sqrt{\varepsilon}\,({}^1/_2\|\eta\|^2 + V(x)) + O(\varepsilon) \tag{7}$$

where the Riemannian metric $\|\ \|$ and the function V are invariant under the action of the torus $\mathbf{T}^n$ on M. The representation of the perturbed Hamiltonian function in the form (7) is the basis of Delon's method in the resonant theory of perturbations /1/.

Without loss of generality one can assume that $F \equiv 0$ on the invariant torus Γ and that the projection of Γ into M under the map $\pi: T^*M \to M$ coincides with the invariant torus Γ_0 of the unperturbed system.

Indeed, suppose Γ is specified by the map $\varphi \in \mathbf{T}^n \mapsto g(\varphi, \sqrt{\varepsilon}) \in M$. We extend the map $\pi\Gamma \to \Gamma_0: \pi g(\varphi, \sqrt{\varepsilon}) \mapsto f(\varphi, x_0)$ to a diffeomorphism $h: M \to M$ which depends smoothly on $\sqrt{\varepsilon}$ and is the identity when $\varepsilon = 0$, and perform a canonical transformation $(x, \eta) \mapsto (h(x, \sqrt{\varepsilon}), h_x^{*-1}\eta)$. In the new variables we then have $\pi\Gamma = \Gamma_0$.

Because the invariant manifolds $\Lambda^\pm$ are Lagrangian and are uniquely projected into M in a neighbourhood of the invariant torus Γ_0, they can be specified by generating functions $S^\pm$, defined independently of ε in a neighbourhood $U \subset M$ of the torus Γ_0:

$$\Lambda^\pm \supset \{(x, \eta) : x \in U, \ \eta = S_x^\pm\}$$

where the functions $S^\pm = S_0^\pm + \sqrt{\varepsilon} S_1^\pm + O(\varepsilon)$ depend smoothly on $\sqrt{\varepsilon}$ and satisfy the Hamilton-Jacobi equations

$$F(x, S_x^\pm, \sqrt{\varepsilon}) = \langle v, S_x^\pm \rangle + \sqrt{\varepsilon}\,(1/2 \, \| S_x^\pm \|^2 + V) + O(\varepsilon) = 0 \tag{8}$$

Hence

$$\langle v, S_{0x}^\pm \rangle = \frac{d}{dt}\Big|_{t=0} S_0^\pm (f(\omega t, x)) = 0 = \langle v, S_{1x}^\pm \rangle + 1/2 \, \| S_{0x}^\pm \|^2 + V$$

It follows from the non-resonance condition (3) on ω that the functions $S_0^\pm$ and $S_1^\pm$ are invariant under the action of $\mathbf{T}^n$ on M:

$$S_{0,1}^\pm (f(\varphi, x)) \equiv S_{0,1}^\pm (x); \ 1/2 \, \| S_{0x}^\pm \|^2 + V \equiv 0 \tag{9}$$

and in addition $\Lambda_0^\pm = \{(x, \eta) : x \in U, \ \eta = S_{0x}^\pm\}$ are invariant manifolds containing the invariant torus Γ_0 of the Hamiltonian system with Hamiltonian function $1/2 \, \|\eta\|^2 + V$. This Hamiltonian system is reversible, so Λ_0^+ turns into Λ_0^- under the reflection $\eta \mapsto -\eta$ and $S_0^- = -S_0^+$. We put $S = 1/2\,(S^+ + S^-)$. Then $S = \sqrt{\varepsilon} S_1 + O(\varepsilon)$, where the function S_1 is invariant under the action of $\mathbf{T}^n$ on M. We will extend S to a function of the form (9) defined on all of M, depending smoothly on $\sqrt{\varepsilon}$, with S_1 invariant with respect to $\mathbf{T}^n$.

Lemma 1. For sufficiently small $\varepsilon > 0$ the function $x \mapsto F(x, S_x, \sqrt{\varepsilon})$ on M reaches a strict non-degenerate maximum equal to zero at $x \in \Gamma_0$.

Proof. Using Eq.(8) and the convexity of F with respect to η,

$$F(x, S_x, \sqrt{\varepsilon}) = F(x, 1/2\,(S_x^+ + S_x^-), \sqrt{\varepsilon}) \leqslant$$
$$1/2\,(F(x, S_x^+, \sqrt{\varepsilon}) + F(x, S_x^-, \sqrt{\varepsilon})) = 0$$

for $x \in U$.

Equality is achieved only if $S_x^+ = S_x^-$, when $x \in \Gamma_0$. Because Λ^+ and Λ^- intersect along Γ at a non-zero angle, the maximum on Γ_0 is non-degenerate

. From the conditions, the function V has a strict maximum equal to zero on the torus Γ_0. Hence $V(x) \geqslant c > 0$ for $x \in M \setminus U$. For $x \in M \setminus U$ we have from (9)

$$F(x, S_x, \sqrt{\varepsilon}) = \langle v, S_x \rangle + \sqrt{\varepsilon}\,(1/2 \, \| S_x \|^2 + V) + O(\varepsilon) =$$
$$\sqrt{\varepsilon} V(x) + O(\varepsilon) \geqslant c\sqrt{\varepsilon} + O(\varepsilon) > 0$$

for sufficiently small $\varepsilon > 0$. The lemma is proved.

We perform a canonical transformation $(x, \eta) \mapsto (x, p)$ where $p = \eta - S_x$. Because $\langle v, S_{1x} \rangle \equiv 0$, in the new variables the Hamiltonian function F preserves its form (7), but the invariant torus Γ of the perturbed system is contained in $M = \{p = 0\}$ and coincides with Γ_0. Furthermore, from Lemma 1 the function $F|_{p=0}$ achieves a strict non-degenerate maximum equal to zero on Γ_0.

We change to the Lagrangian form of the equations of motion, performing a Legendre transformation

$$(x, p) \mapsto (x, x^\cdot) \in TM; \ x^\cdot = F_p \tag{10}$$
$$L(x, x^\cdot, \sqrt{\varepsilon}) = \max_p (\langle p, x^\cdot \rangle - F(x, p, \sqrt{\varepsilon}))$$

By virtue of (6) the function F is convex with respect to the momentum p, and so the transformation (10) is well-defined and the function L is convex with respect to the velocity $x^\cdot$:

$$\frac{\|\xi\|^2}{b\sqrt{\varepsilon}} \leqslant \langle L_{x^\cdot x^\cdot}\xi, \xi \rangle \leqslant \frac{\|\xi\|^2}{a\sqrt{\varepsilon}} \tag{11}$$

for all $\xi \in T_xM$, where $\|\xi\|^2 = \langle \xi, A^{-1}(x)\xi\rangle$. Explicitly,

$$L = \frac{1}{2\sqrt{\varepsilon}}\|x^{\cdot} - v(x)\|^2 - \sqrt{\varepsilon}V(x) + \varepsilon R\left(x, \frac{x^{\cdot} - v(x)}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}\right) \tag{12}$$

By virtue of (10) $\min_{x^{\cdot}} L(x, x^{\cdot}, \sqrt{\varepsilon}) = -F(x, 0, \sqrt{\varepsilon})$, so that the function L on TM reaches a strict non-degenerate minimum equal to zero on the invariant torus Γ_0, i.e., with $x \in \Gamma_0$ and $x^{\cdot} = v(x)$.

The subsequent part of the proof of the theorem is performed using the methods of global variational calculus and follows /6/. Let Ω be the set of completely continuous curves γ: $[0, \tau] \to M$, such that

$$\int_0^{\tau} \|\gamma^{\cdot}(t)\|^2 dt < \infty$$

and $\gamma(0), \gamma(\tau) \in \Gamma_0$. (The length $\tau(\gamma) > 0$ of the segment $[0, \tau]$ depends on the curve γ). For a given $h > 0$ we define Hamilton's action functional S_h on Ω by the formula

$$S_h(\gamma) = \int_0^{\tau} (L(\gamma(t), \gamma^{\cdot}(t), \sqrt{\varepsilon}) + \sqrt{\varepsilon}h)\, dt \tag{13}$$

If γ is a critical point of the functional (13) then $\gamma(t)$ is a solution of Lagrange's equations, and from the formula for the variation of the action functional

$$\delta S_h(\gamma) = \langle p(t), \delta\gamma(t)\rangle |_0^{\tau} + (h\sqrt{\varepsilon} - F(\tau))\,\delta\tau$$

$$p(t) = L_{x^{\cdot}}(\gamma(t), \gamma^{\cdot}(t), \sqrt{\varepsilon}),\ F(t) = F(\gamma(t), p(t), \sqrt{\varepsilon})$$

Hence $\langle p(0), \xi\rangle = 0$ for all $\xi \in T_{\gamma(0)}\Gamma_0$, $\langle p(\tau), \xi\rangle = 0$ for all $\xi \in T_{\gamma(\tau)}\Gamma_0$ and $F(t) \equiv h\sqrt{\varepsilon}$. By virtue of (6)

$$h\sqrt{\varepsilon} = F(\gamma(0), p(0), \sqrt{\varepsilon}) \geqslant \langle p(0), v(\gamma(0))\rangle + {}^1\!/_2 a\sqrt{\varepsilon}\|p(0)\|^2 = {}^1\!/_2 a\sqrt{\varepsilon}\|p(0)\|^2$$

(using $F_p(x, 0, \sqrt{\varepsilon}) = v(x) \in T_x\Gamma_0$ for $x \in \Gamma_0$). Consequently,

$$\|p(0)\|^2 \leqslant 2h/a,\ \|p(\tau)\|^2 \leqslant 2h/a \tag{14}$$

From (6) and (14) we obtain

$$\|\gamma^{\cdot}(0) - v(\gamma(0))\| = \|F_p(\gamma(0), p(0), \sqrt{\varepsilon}) - F_p(\gamma(0), 0, \sqrt{\varepsilon})\| \leqslant b\sqrt{\varepsilon}\|p(0)\| \leqslant b\sqrt{2\varepsilon h/a} \tag{15}$$

It follows from (15) that the departure time for the trajectory $\gamma(t)$ from the neighbourhood U of the invariant torus Γ_0 is

$$t_- = \inf\{t \in [0, \tau] : \gamma(t) \notin U\} \geqslant C/\sqrt{\varepsilon h} \tag{16}$$

Similarly, the arrival time into U is

$$t_+ = \sup\{t \in [0, \tau] : \gamma(t) \notin U\} \leqslant \tau - C/\sqrt{\varepsilon h} \tag{17}$$

where $C > 0$ is independent of ε and h.

Lemma 2. There exists $C > 0$ such that for all sufficiently small $\varepsilon > 0$ and $h > 0$ the functional S_h has a critical point $\gamma \in \Omega$ such that $\gamma([0, \tau]) \not\subset U$ and $S_h(\gamma) \leqslant C$.

We will confine ourselves to an outline of the proof, because it is close to an assertion proved in /6/. We define on Ω the structure of a Hilbert manifold /7/. Because of the quadratic behaviour of the function L with respect to $x^{\cdot}$ as $\|x^{\cdot}\| \to \infty$ one can conclude /7/ that S_h is a function of class $C^{1+\mathrm{Lip}}$ on Ω. Let $\Omega(U)$ be the set of curves from Ω completely contained in U. The functional S_h on Ω does not satisfy "condition C" of Palais-Smale /7/, and so Morse's theory is not directly applicable. From the positivity of L on the domain $M \setminus \Gamma_0$ and inequality (11) one can conclude that $S_h(\gamma) \to +\infty$ for $\gamma \in \Omega \setminus \Omega(U)$ and $\tau(\gamma) \to 0$ or $\tau(\gamma) \to +\infty$. From this it follows /6/ that for any $c > 0$ the functional S_h satisfies an analogue of "condition C" on the complete subset $\Omega^c \setminus \Omega(U)$, where $\Omega^c = \{\gamma \in \Omega\colon S_h(\gamma) \leqslant c\}$. From this and from the non-degeneracy of the minimum of the function L on Γ_0 one can conclude that if the functional (13) has no critical points on the set $\Omega^c \setminus \Omega(U)$, then there exists a one-parameter semigroup of action-reducing continuous transformations of the set Ω^c, contracting Ω^c to $\Omega(U)$. (The proof is based on the method of gradient descent, see /6/).

Thus to prove the lemma it is sufficient to construct a continuous map $\psi: S^k \to \Omega$ of a k-dimensional sphere into Ω, not contractible to $\Omega(U)$ and such that

$$\max_{\psi(S^k)} S_h \leqslant C$$

where $C > 0$ does not depend on ε and h.

Let $\pi: M \to N$ be the projection onto the compact quotient manifold $N = M/T^n$. Then for some $k \geqslant 0$ there exists a smooth map $S^{k+1} \to N$ taking a pole of the sphere S^{k+1} into the point $\pi(\Gamma_0)$ and not contractible to $\pi(U)$ /8/. For every point on the equatorial sphere $S^k \subset S^{k+1}$ the meridian passing through it defines a closed curve in N. We lift this curve to a horizontal curve in M with endpoints in Γ_0. (A curve in M is horizontal /8/ if at every point its velocity vector is orthogonal to the fibres of the bundle $\pi: M \to N$ in the metric $\| \ \|$.) We obtain a map $\psi_0: S^k \to \Omega$ that is not contractible to $\Omega(U)$. One can assume that $\psi_0(S^k)$ does not contain single-point curves and reparameterize each curve $\gamma_0 \in \psi_0(S^k)$ so that $\|\gamma_0^{\cdot}(t)\|^2 \equiv \varepsilon$. Because S^k is compact the lengths of the curves from $\psi_0(S^k)$ are bounded:

$$\int_0^{\tau(\gamma_0)} \|\gamma_0^{\cdot}(t)\| \, dt = \tau\sqrt{\varepsilon} \leqslant A \tag{18}$$

We associate the curve $\gamma_0 \in \psi_0(S^k)$ with the curve $t \mapsto \gamma(t) = f(\omega t, \gamma_0(t))$ from Ω. The resulting map $\psi: S^k \to \Omega$ is not contractible to $\Omega(U)$. For $\gamma \in \psi(S^k)$ we have, by virtue of (12) and (18),

$$S_h(\gamma) = \int_0^{\tau} \left[\frac{1}{2\sqrt{\varepsilon}} \|\gamma^{\cdot} - v(\gamma)\|^2 + \sqrt{\varepsilon}(h - V(\gamma)) + \varepsilon R\left(\gamma, \frac{\gamma^{\cdot} - v(\gamma)}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}\right) \right] dt =$$

$$\int_0^{\tau} \left[\frac{1}{2\sqrt{\varepsilon}} \|\gamma_0^{\cdot}\|^2 + \sqrt{\varepsilon}(h - V(\gamma_0)) + \varepsilon R\left(\gamma, f_x(\omega t, \gamma_0)\frac{\gamma_0^{\cdot}}{\sqrt{\varepsilon}}, \sqrt{\varepsilon}\right) \right] dt \leqslant$$

$$B\sqrt{\varepsilon}\tau \leqslant AB = C \quad (\gamma = \gamma(t), \ \gamma_0 = \gamma_0(t))$$

The lemma is proved.

Let $h \to +0$, and let $\gamma_h \in \Omega$ be the family of trajectories constructed in Lemma 2. Because $\gamma_h \not\subset U$, by virtue of (16) and (17) one can assume that $\gamma_h: [a, b] \to M$ for $a < 0 < b$ and $\gamma_h(0) \notin U$, and $a(h) \to -\infty$ and $b(h) \to +\infty$ for $h \to +0$. Because the equality $F \equiv h\sqrt{\varepsilon}$ is satisfied on the trajectory γ_h and the function L is convex with respect to velocity, the norm $\|\gamma_h^{\cdot}(0)\|$ is uniformly bounded and, consequently, one can find a sequence $h_i \to +0$ such that there exist $\lim \gamma_{h_i}(0) = x^0 \notin U$ and $\lim \gamma_{h_i}^{\cdot}(0) = v^0$ as $i \to \infty$. Let $t \to x(t)$ be the solution of Lagrange's equations with initial conditions $x(0) = x^0$ and $x^{\cdot}(0) = v^0$. Then the trajectory $x(t)$ is infinitely extendible and

$$\int_{-\infty}^{+\infty} L(x, x^{\cdot}, \sqrt{\varepsilon}) \, dt = \lim_{i \to \infty} \int_{a(h_i)}^{b(h_i)} L(\gamma_{h_i}, \gamma_{h_i}^{\cdot}, \sqrt{\varepsilon}) \, dt \leqslant C \tag{19}$$

exists.

Because $L \geqslant 0$ and $L = 0$ only when $x \in \Gamma_0$ and $x^{\cdot} = v(x)$, the trajectory $t \to x(t)$ is doubly-asymptotic to Γ_0 for $t \to \pm\infty$.

In order to conclude the proof of the theorem it remains to show that for sufficiently small $\varepsilon > 0$ the phase space trajectory $t \to (x(t), p(t))$ corresponding to $x(t)$ is contained in a domain $\{\|p\| < A\}$, where $A > 0$ does not depend on ε. From the form of the Lagrangian function (12) it follows that $\|x^{\cdot}(t) - v(x(t))\|^2 \leqslant B\varepsilon$ for $L(x(t), x^{\cdot}(t), \sqrt{\varepsilon}) \leqslant \sqrt{\varepsilon}$, and, consequently,

$$\|p(t)\|^2 = \|x^{\cdot}(t) - v(x(t))\|^2 / \varepsilon + O(\sqrt{\varepsilon}) \leqslant B + O(\sqrt{\varepsilon}) \tag{20}$$

From (19), the time during which the inequality $L(x(t), x^{\cdot}(t), \sqrt{\varepsilon}) \leqslant \sqrt{\varepsilon}$ is not satisfied is no greater than $C/\sqrt{\varepsilon}$. From (7) it follows that

$$\|p(t)\|^{\cdot} = \{{}^1\!/_2\|p\|^2, F\} / \|p\| = (\sqrt{\varepsilon}\langle Ap, V_x\rangle + \{\|p\|^2, O(\varepsilon)\}) / \|p\| \leqslant \sqrt{\varepsilon}\|V_x\| + O(\varepsilon)$$

From this, using (20) we obtain

$$\|p(t)\| \leqslant \sqrt{B} + C \max \|V_x\| + O(\sqrt{\varepsilon}) \leqslant A$$

for $-\infty < t < \infty$, which is what required. The theorem is proved.

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CONTROL OF THE SPEED OF RESPONSE OF PREDATOR-PREY SYSTEMS*

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The problem of optimal of a predator-prey system is investigated. The existence of admissible control is established and the structure of optimal control is investigated.

Problems of optimal control of biological communities have been studied in many papers; bibliographies are contained in /1, 2/.

1. Statement of the problem. The dynamics of the interaction of predators and prey are described by the equation /3/

$$x_1^{\cdot}(\tau) = (a_1 - a_2 y_1) x_1, \; y_1^{\cdot}(\tau) = (a_3 x_1 - a_4) y_1 \tag{1.1}$$

where $x_1(\tau)$ is the population density of the prey and $y_1(\tau)$ that of the predators at time τ, and a_i are positive numbers characterizing the interspecific interactions.

In practice, to influence the system purposefully, one uses various chemical preparations such as pesticides, which act only on the prey, or only on the predators, or on both populations simultaneously.

First we will study the situation in which the control acts only on the prey. For the remaining two cases we restrict ourselves to describing the final result.

We will change to dimensionless variables given by the formulae

$$x_1(\tau) = a_4 a_3^{-1} x(t), \; y_1(\tau) = a_1 a_2^{-1} y(t), \; b = a_4 a_1^{-1}, \; \tau = a_1 t$$

Using the dimensionless variables in (1.1), the equations of the controlled system have the form

$$x^{\cdot}(t) = (1 - y) x - ux, \; y^{\cdot}(t) = b(x - 1) y \tag{1.2}$$

$$x(0) = x_0, \; y(0) = y_0, \; x_0 > 0, \; y_0 > 0, \; t \geqslant 0 \tag{1.3}$$

The control $u(t)$ satisfies the natural constraints

$$0 \leqslant u \leqslant \gamma, \; \gamma = \text{const} > 0 \tag{1.4}$$

For $u = 0$, system (1.2) has two equilibrium positions in the x, y plane: the points $(0, 0)$ and $(1, 1) = R$. Because only the point R is of any actual interest, the controllers' objective is to take system (1.2) from an arbitrary initial position (x_0, y_0) to the position R in the least possible time. Thus if $T(x_0, y_0, u)$ is the instant when system (1.2) first reaches the point R,

*Prikl. Matem. Mekhan., 54, 3, 502-506, 1990